A complex variable boundary-element strategy for determining groundwater flow nets and travel times

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Received 19 August 2002; received in revised form 2 December 2002; accepted 3 December 2002

Abstract

The complex variable boundary-element method is routinely used to determine the complex potential, \( \theta = \phi + i\psi \), at any position, \( z = x + iy \), internal to the flow domain. We reverse this mapping by exactly determining the complex position for specified complex potentials—in effect solving for \( z(\theta) \) instead of \( \theta(z) \). Direct calculation of the physical location, \( [x, y] \), of potential, \( [\phi, \psi] \), intersections greatly simplifies the determination of groundwater flow nets and travel times. One problem arises when overlapping sheets in the complex potential domain form due to multiple capture zones, with dividing stream lines forming branch cuts. We avoid this problem by resolving individual capture zones, and then determining the flownet within each zone. Travel times are readily calculated using increments in potential along a streamline. Examples of flow under a dam, free surface delineation within a dam, regional flow, and dipole flow are used to demonstrate the method’s utility.

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Keywords: Flownet; Travel time; Complex variables; Boundary integrals; Ordinary least squares

1. Introduction

A flownet is a traditional and valuable tool that aids in the analysis and interpretation of groundwater flow and contaminant transport problems. Flownets help to visualize the flowfield, to delineate the capture zone or influence areas of boundary discharge and recharge, to design groundwater development and remediation measures, and to evaluate the effects of alternative boundary conditions during site characterization. For transport problems, flowpaths and travel times found by particle tracking are used to identify outflow locations and the arrival time of contaminants, as well as to show the advective advance of a contaminant front within the flow domain.

Two methods are available for determining the flow field. One method utilizes potential and stream functions to identify the locations of the equipotential and streamlines, i.e., the flownet. Because most numerical procedures provide only the values of the potential and stream function at a limited number of locations, interpolation using a contouring algorithm is usually necessary. Interpolation procedures can introduce additional errors into the flownet geometry delineation. The second method utilizes knowledge of the velocity field to construct pathlines. Because the velocity field is usually deduced from the results of flow modeling, the accuracy of the flow paths depends on both the flow model and the interpolation scheme used to construct the velocity field. Apart from the accuracy of the velocity field, the accuracy of calculated travel times is also related to the step size associated with the time-stepping scheme. Fine discretization in both space and time are major constraints to the application of both flownet and pathline methods to travel-time calculations. Pathlines can be obtained for block-centered, finite-difference models using linear interpolation of the flow within each cell [1]. This approach finds pathlines within a finite-difference grid cell using an analytical function of nodal values. While the velocity component is continuous in the direction of flow, discontinuities arise in the direction transverse to flow due to the need to discretize the domain into grid cells. A continuous velocity field may be obtained using the finite-element method [2], which
avoids the transverse direction discontinuity by using a bilinear interpolation for both head and stream functions. Both finite-difference and finite-element methods provide pathlines and travel times, and eliminate the need for time stepping. The shortcoming of both techniques, however, is the need for internal discretization of the flow domain.

Internal discretization of the flow domain can be avoided for conditions of steady flow through homogeneous media using real-variable boundary-element methods, complex-variable boundary-element methods (CVBEM), and analytic element methods (AEM). The advantage of AEM and CVBEM lies in their ability to determine both stream potentials and velocities at points internal to the flow domain with excellent accuracy and continuity, laying the foundation upon which contaminant transport problems can be addressed. While generally restricted to problems of steady flow, AEM and CVBEM can still be applied to long-term contaminant transport problems if short-term transients are unimportant and the steady flow assumption is acceptable [3]. Nonetheless, these methods need to perform either a time-stepping or contour interpolation when plotting flownets and calculating travel times.

In this paper, we present a new method for flownet determination and pathline computation by introducing a CVBEM post-processing strategy that provides the flownet geometry. Instead of finding the value of the complex potential, \( \theta = \phi + i \psi \), for given complex positions, \( z = x + iy \), in the physical plane, our method reverses this procedure. We directly identify the positions of streamline–equipotential intersections using complex boundary locations and potentials.

For cases where multiple capture zones induce branch cuts (i.e., complex positions overlap in the complex potential domain), regions of one-to-one correspondence are found using a capture zone delineation procedure [4]. The delineation procedure can be applied in either an upstream or downstream direction, and is especially useful when trying to determine multiple capture zones.

Our reverse-mapping approach also provides travel times. Instead of determining travel times by summing travel-time increments between selective locations, \( \Delta t \), along a streamline we obtain travel times using increments in fluid potential, \( \Delta \phi \), along each streamline. An example computation shows that the computed flownet, solute fronts, and travel times are in excellent agreement with an analytical solution.

2. Problem formulation

Two-dimensional, steady, groundwater flow in a homogeneous, isotropic medium without internal sources and sinks is governed by the Laplace equation:

\[
\nabla \cdot q = \nabla^2 u = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{1}
\]

where \( q = \nabla h = Q/A \) is the flux vector, or flow \( Q \), per unit cross-sectional area, \( A \), \( h \) is the principal unknown, which can be either potential, \( \phi \) (defined here as the negative product of the total head with the hydraulic conductivity), the stream function, \( \psi \), or the complex potential, \( \theta = \phi + i \psi \), where \( i \) is the imaginary unit. The CVBEM provides a method for simultaneously obtaining the solution of the potential and stream function [5–7]. Most CVBEMs employ the Cauchy integral using complex variables:

\[
\theta[z] = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{\theta[\zeta]}{z - \zeta} \, d\zeta \tag{2}
\]

where \( z = x + iy \) is the complex position and \( \Gamma \) is the boundary of the flow domain. This boundary integral equation is reduced to an algebraic system of linear equations using prescribed boundary potentials and geometries. If all complex potential boundary conditions can be defined, then the complex potential internal to the domain is calculated using

\[
\theta[z] = \frac{1}{2\pi i} \sum_{k=1}^{n} \left[ \theta_k \left( \frac{z_{k+1} - z}{z_{k+1} - z_k} \right) - \theta_{k+1} \left( \frac{z_k - z}{z_{k+1} - z_k} \right) \right] \ln \left( \frac{z_{k+1} - z}{z_k - z} \right) \tag{3}
\]

where \( n \) is the number of boundary elements, and linear interpolation of \( \theta \) along \( \Gamma \) has been used:

\[
\theta[z] = \theta_k \left( \frac{z_{k+1} - z}{z_{k+1} - z_k} \right) - \theta_{k+1} \left( \frac{z_k - z}{z_{k+1} - z_k} \right) \tag{4}
\]

and where the subscripts \( k \) and \( k + 1 \) are the numberings of the end-nodes in element \( k \), \( k + 1 \), \( \theta_k \) and \( \theta_{k+1} \) are the corresponding nodal values of the complex potential, and \( z_k \) and \( z_{k+1} \) are the corresponding complex nodal positions.

The flux vector, \( q \), can be written in terms of the derivatives of either fluid potential, \( \phi \), or stream potential, \( \psi \):

\[
q = q_x + iq_y = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial x} \tag{5}
\]

where \( q_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \) and \( q_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \) are the fluid fluxes in the \( x \)- and \( y \)-directions, respectively. These relationships are known as the Cauchy–Riemann equations. The flux vector can be related to the first derivative of the complex potential:

\[
\omega = \frac{d\theta}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} - i \frac{\partial \phi}{\partial y} = \overline{q} \tag{6}
\]

where \( \overline{q} \) is the complex conjugate of the flux vector.

A significant advantage of CVBEM (and AEM) lies in the ability to represent potential and stream functions as continuously differentiable functions. That is, the
potential and its derivatives vary smoothly within the flow domain. The first derivative of the complex potential is deduced from the Cauchy integral theorem:

\[ \omega(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} g_k \theta_k \]

where

\[ g_k = \left[ \frac{1}{z_k - z_{k-1}} \right] \ln \left[ \frac{z_k - z}{z_{k-1} - z} \right] - \left[ \frac{1}{z_{k+1} - z_k} \right] \ln \left[ \frac{z_{k+1} - z}{z_k - z} \right] \]

and where linear interpolation of \( \theta \) along \( \Gamma \) has again been used.

3. Flownet solution strategy

We note that a complex potential problem in the physical plane, \( \theta[z] \), can be transformed into a complex position problem in the complex potential plane, \( z[\theta] \). The expressions in the physical plane are

\[ \phi = \phi(x,y) \]
\[ \psi = \psi(x,y) \]

These can be regarded as a transform from the physical plane, \( z = x + iy \), to the complex potential plane, \( \theta = \phi + i\psi \). The Jacobian of the transform, \( J_b \), is given by

\[ J_b = \text{DET} \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{bmatrix} = q_x^2 + q_y^2 = ||q||^2 \]

The differentiation rule on a compound function, \( f(\phi(x,y), \psi(x,y)) \), leads to

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial x} \]
\[ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial y} \]

By applying the differentiation rule to \( x \) and \( y \), we obtain, respectively

\[ \frac{\partial \phi}{\partial x} = \frac{1}{||q||} \left[ \frac{\partial \phi}{\partial \psi} \right] \]
\[ \frac{\partial \psi}{\partial x} = \frac{1}{||q||^2} \left[ -\frac{\partial \phi}{\partial \psi} \right] \]
\[ \frac{\partial \phi}{\partial y} = \frac{1}{||q||} \left[ -\frac{\partial \psi}{\partial \psi} \right] \]
\[ \frac{\partial \psi}{\partial y} = \frac{1}{||q||^2} \left[ \frac{\partial \phi}{\partial \psi} \right] \]

If the fluid flux can be determined, then it follows that:

\[ \frac{\partial \phi}{\partial \phi} = \frac{\partial \psi}{\partial \psi} = q_x \]
\[ \frac{\partial \phi}{\partial \phi} = \frac{\partial \psi}{\partial \psi} = q_y \]
\[ \frac{\partial \psi}{\partial \phi} = \frac{\partial \psi}{\partial \psi} = \frac{\partial \phi}{\partial \phi} = 0 \]

These equations are the Cauchy–Riemann conditions and governing equations in the complex potential plane, respectively. Because the governing equations can be written as

\[ \frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial \psi^2} = 0 \]

we can interchange the complex potential with the complex position in the solution presented as Eq. (3):

\[ z[\theta] = \frac{1}{2\pi i} \sum_{k=1}^{n} \left[ \frac{\theta_{k+1} - \theta_k}{z_k - z} \right] \ln \frac{z_{k+1} - z_k}{z_k - z} \]

so that we can find the complex position for a specified complex potential within a flow domain. Solution of the governing equations requires that all complex potential boundary conditions be defined, which can be accomplished by first solving the boundary value problem in the physical plane.

Linear interpolators for both the complex position and complex potential problems result in mutual linearity of \( \theta[z] \) with \( z[\theta] \). This can be seen from

\[ \theta[z] = \frac{\theta_1(z_2 - z) + \theta_2(z - z_1)}{z_2 - z_1} \]

Solving for \( z \) results in

\[ z[\theta] = z_1(\theta_2 - \theta) + z_2(\theta_1 - \theta) \]

These equations ignore error in the interpolation function. When a complex potential interpolation error, \( \epsilon \), is present, then these equations become

\[ \theta[z] = \frac{\theta_1(z_2 - z) + \theta_2(z - z_1)}{z_2 - z_1} + \epsilon \]

Solving for \( z \) results in

\[ z[\theta] = z_1(\theta_2 - \theta) + z_2(\theta - \theta_1) + \epsilon \]

where

\[ \epsilon = z_1 - z_2 \theta \]

is the position interpolation error. This error analysis demonstrates that uncertainties or errors in potentials affect interpolated positions, and that reductions in
boundary potential uncertainties can improve predicted positions.

This error analysis is not intended to suggest that the reverse-mapping technique is limited to linear boundary interpolation schemes. Quite the contrary, any set of higher-order boundary interpolation schemes can be used in the formulation of both the forward and reverse problems. Our intent here is to emphasize the need for accurate specification of the boundary values in order to obtain an accurate flownet.

4. Applications

As an example of the use of this reverse-mapping strategy, Fig. 1 presents the problem of groundwater

![Fig. 1. Flow of water under a dam with a sheet pile. (a) Profile view of physical domain: solid lines are streamlines, ψ, dashed lines are equipotentials, ϕ. (b) Complex potential domain: solid lines are constant elevation, y, dashed lines are constant horizontal distance, x. Numbers are nodal positions.](image)
flow beneath a dam containing a sheet pile [8]. The calculated flownet shows the positions of streamline–equipotential intersections. Only the right half of the flow field is presented because the problem is symmetric. A total of 50 nodes are used to discretize the boundary, with the greatest nodal density near the bottom of the sheet pile. Prescribed heads are assigned along the upstream (between nodes 17 and 27) and downstream (nodes 43–1) surfaces. No-flow boundaries are prescribed along the dam and sheet pile (nodes 1–17), and along the bottom of the domain (nodes 27–38). Boundary conditions along the right boundary (nodes 39–42) are not specified.

An overdetermined system of equations results because the complex variable formulation provides two equations (real and imaginary) for each node [9,10]. Overdetermined systems of equations are readily solved using ordinary least squares (OLS) [11]. OLS is especially useful for determining the unknown potential of nodes along the right-hand boundary. In this case, the problem becomes one of finding the values of unknown boundary values which uniquely minimizes the estimation errors at nodes where boundary values are known. While leaving the right boundary undefined clearly makes the problem ill-posed, the resulting solution is unique, and provides the optimal fit to the imposed boundary conditions. Note that the estimated complex potential forms an arc, which is consistent with a logarithmic singularity at infinity.

OLS can also be used to determine the position of boundary nodes, such as along a free surface [12]. Fig. 2 presents the problem of flow through a porous dam with a seepage face. A total of 53 nodes are used. Prescribed constant head boundary conditions are assigned along the submerged portions of the upstream and downstream faces (nodes 1–13 and 23–33). Heads along the seepage face equal the nodal elevation (nodes 33–39). Prescribed no-flux segments are assigned along the bottom boundary (13–23) and upper free surface (40–1). Elevations of the free surface are iteratively adjusted until the calculated total head equals their elevation. This is accomplished by specifying an initial elevation and solving for the unknown head. Nodes 41–53 are then moved vertically so that the elevation equals the calculated head. Node 40 is shifted along the dam face until the elevation equals the calculated head.

\begin{equation}
\text{d}\phi \text{d}\psi = \left[ \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial x} \right] \text{d}x \text{d}y = J_b \text{d}x \text{d}y
\end{equation}

represents the ratio between the differential areas in the complex potential and physical planes. If the Jacobian or its reciprocal vanishes at some point in the physical plane, then an infinitesimal region in one plane collapses into a single point in the other, which results in a violation of the one-to-one correspondence criterion for a transform. Because the Jacobian, \( J_b = ||q||^2 \), in our transform, this violation arises at stagnation points, sources, and sinks. For our proposed method to apply, we need to exclude these conditions from the domain interior.

While a nonzero Jacobian is necessary to maintain the one-to-one correspondence of a transform locally, a nonzero Jacobian throughout a domain does not necessarily guarantee a one-to-one transform in the entire domain. A counterexample has been shown [13] in which \( u = x \cos y \), \( v = x \sin y \), and \( J_b = x \). For this example, the Jacobian is positive in the right half plane, yet the transform is not one-to-one in the half plane because \((a, b + n\pi)\) where \( n = 0, 1, 2, \ldots \), have the same image. If a transform is locally one-to-one and certain conditions are satisfied within the complex potential and physical planes, one can infer the transform is one-to-one in the entire domain. One such condition requires the domain in the complex potential plane to be simply connected [13]. We note that the Jacobian may vanish only at points along the boundary. This is because both and are continuous, differentiable, harmonic functions, and because extreme values for harmonic functions can arise only along the boundary. Because the Jacobian is always positive within the domain, the transform is orientation-preserving; i.e., a particle rotates through a simple closed curve in the physical plane and in the complex potential plane in the same direction.

The one-to-one correspondence suggests a single-valued property of the respective dependent variables in both planes. The single-valued property of both \( \phi \) and \( \psi \) in the physical plane is assured if the singularities such as sources and sinks have been excluded. We can assure the single-valued property of \( \phi \) and \( \psi \) in the complex potential plane in several ways. The most straightforward way is to plot the boundary image in the complex potential plane using the complex potentials computed on the boundary in the physical plane and check the image of the domain enclosed by the boundary.

We sometimes find that images of boundary segments overlap to form multiple sheets with one or more branch cuts in the complex potential plane, even though the domain image remains simply connected. In this case, the one-to-one correspondence, or the single-valued property, in the domain holds. When the image domain has multiple sheets, we must separate the domain into
subdomains where the one-to-one correspondence remains. We do this by examination of the branch cuts that connect the sheets. In this case, locating stagnation points along the external boundary is helpful. The stagnation points are bifurcations at which dividing streamlines may originate or terminate. Once the dividing streamline has been located and the subdomains determined, we can use the strategy in the previous section within each subdomain.

To demonstrate a possible pitfall associated with a failure to maintain the single-valued property, Fig. 3–5 present a regional flow problem using a profile view in the physical plane. A total of 58 nodes are used. The upper surface (between nodes 37 and 1) in all figures consists of a specified head equal to the elevation. In Fig. 3, the right, left, and lower surfaces (between nodes 1 and 37) are no-flow boundaries. In Fig. 4, a unit inflow flux is prescribed along a length of the left boundary (between nodes 7 and 12). In Fig. 5, a unit outflow flux is prescribed along a length of the right boundary (between nodes 26 and 31). Note that the single-valued property of \( x \) and \( y \) in the complex po-
The complex potential plane is destroyed for the problem shown in Fig. 5 because of the multiple sheets. The subdomains meeting the one-to-one correspondence must first be located before using Eq. (21) to find the position of the specified flownet.

To resolve the problem of overlapping positions within the complex potential plane, we subdivide the physical flow domain into separate flow subdomains using a dividing streamline to maintain the single-valued property of $x$ and $y$ in their images, respectively. The position of the streamline along the flow divide is found by starting at the stagnation point on the boundary where the dividing streamline originates or terminates, and then a first-order approximation of the trajectory is used to track the dividing streamline position [4]. One advantage of our approach is that errors in the position of successive points along the divide are never accumulated. Also, the procedure can proceed either upstream or downstream. The method can also be used for locating an equipotential line.

To illustrate the use of dividing streamlines, the image in the complex potential plane (Fig. 5(b)) is subdivided into two subdomains using a dividing streamline, shown as Fig. 6. The resulting flownet is presented as Fig. 5(a). The dividing streamline is found using the prediction–correction method described above. The dividing streamline delimits the two domains, the capture zone of the lower drain and the stream capture zone.

![Fig. 3. Profile view of regional flow to a stream with free replenishment on upper surface only: (a) profile view of physical domain; (b) complex potential domain.](image-url)
6. Travel-time computation

The travel time, \( \tau \), of a fluid particle moving from point \( A \) to point \( B \) along a streamline is the path integral of the inverse velocity:

\[
\tau = \int_A^B \frac{1}{v} \, ds
\]  

(28)

where \( s \) is distance along the streamline, and the fluid (or particle) velocity, \( v \), at any point internal to a flow domain is

\[
v = v_x + iv_y = \frac{q}{n_e} = \frac{\overline{\omega}}{n_e} 
\]  

(29)

where \( v_x \) and \( v_y \) are the components of the velocity vector in the \( x \)- and \( y \)-direction, respectively, \( n_e \) is the effective porosity, and \( \overline{\omega} \) indicates the complex conjugate of \( \omega \). The travel time can now be determined using

\[
\tau = \frac{1}{n_e} \int_{\phi_1}^{\phi_2} \frac{1}{\|v\|^2} \, d\phi 
\]  

(30)

because \( \|v\|^2 = v^r \) and \( \partial \psi / \partial s = 0 \) along a streamline so that

\[
\tau = \frac{\omega}{n_e} = \frac{1}{n_e} \frac{\partial \psi}{\partial s} 
\]  

(31)

The travel time can now be written as

\[
\tau = n_e \int_{\phi_1}^{\phi_2} \|u\|^2 \, d\phi 
\]  

(32)

where

Fig. 4. Regional flow to a stream with additional inflow source on lower left surface of physical domain: (a) profile view of physical domain; (b) complex potential domain.
\[
u = \frac{dz}{d\phi} = \frac{\partial x}{\partial \phi} + i \frac{\partial y}{\partial \phi} = \frac{q_x + iq_y}{J_b} = \frac{1}{\omega}
\]

and

\[
\frac{1}{||v||^2} = n_x^2 ||u||^2
\]

We now have an equation that predicts the travel time of a particle moving from point \( A \) to point \( B \) along a streamline in the complex potential plane. The integration is readily performed using quadrature by transforming the integral interval \( \phi \in [\phi_B, \phi_A] \) to \( \zeta \in [-1, 1] \):

\[
\phi = \frac{\phi_B - \phi_A}{2} \zeta + \frac{\phi_B + \phi_A}{2}
\]

yielding

\[
\tau = \frac{(\phi_B - \phi_A)n_e}{2} \int_{-1}^{1} ||w||^2 d\zeta
\]

Gaussian quadrature leads to

\[
\tau = \frac{\Delta \phi n_e}{2} \sum_j G_j ||w||^2_j
\]

where \( \Delta \phi = \phi_B - \phi_A \) is the prescribed potential increment, \( j \) indicates the \( j \)-th quadrature point, and \( G_j \) and \( ||w||^2_j \) are, respectively, the weighting factor and integrand values at point \( \zeta_j \). The values of \( \psi \) and \( \Delta \phi \) are normally prescribed while the values of \( u \) are calculated using the complex potential analog to the complex position derivative:

Fig. 5. Regional flow to a stream with additional outflow sink on lower right surface of physical domain: (a) profile view of physical domain; (b) complex potential domain.
where

\[ f_k = \left[ \frac{1}{\theta_k - \theta_{k-1}} \right] \ln \left[ \frac{\theta_k - \theta}{\theta_{k-1} - \theta} \right] - \left[ \frac{1}{\theta_{k+1} - \theta_k} \right] \ln \left[ \frac{\theta_{k+1} - \theta}{\theta_k - \theta} \right] \]

\[ u[\theta] = \frac{dz}{d\theta} = \frac{\partial x}{\partial \phi} i + \frac{\partial y}{\partial \phi} = \frac{1}{2\pi i} \sum_{k=1}^{n} f_k z_k \]

7. Sources and sinks

The effects of sources and sinks can be included by noting that flow is governed by a linear equation, together with linear boundary conditions, so that the principle of superposition applies. Provided that the strength of the source or sink is given, we can subtract the boundary values induced by the sources or sinks from the original boundary conditions, and then solve the problem with the remaining boundary conditions.
The final solution provides the summation of the sources or sinks along with the remainder [6,14].

The complex potential method presented above can also be used for problems containing sources and sinks. The only difference from problems with no sources and sinks is that, after the boundary data have been obtained in the physical plane by the CVBEM, we need to exclude the sources and sinks from the original domain by introducing cutting lines so that the new domain(s) contains no sources and sinks. The complex potential values on the cutting lines can be computed from the original problem solution, these values constitute a part of the boundary data of the new domain(s) where we will use the complex potential algorithm.

To demonstrate this application, we examine the dipole problem which consists of a source and a sink, each with a unit strength. The source and sink are separated by a distance of two units in a domain with no ambient flow. One reason for choosing such an example is the availability of an exact analytical solution for the travel time [15]:

\[
\tau = \frac{d^2}{q \sin^2 \xi} \left( \frac{\sin \eta}{\cosh \eta + \cos \xi} - \frac{2}{\tan \xi} \tan^{-1} \left[ \tan \frac{\xi}{2} \tan \frac{\eta}{2} \right] \right)
\]

\[
- \frac{d^2}{q \sin^2 \xi} \left( \frac{\sin \eta_o}{\cosh \eta_o + \cos \xi} - \frac{2}{\tan \xi} \tan^{-1} \left[ \tan \frac{\xi}{2} \tanh \frac{\eta_o}{2} \right] \right)
\]

where \( q \) is the fluid flux, \( d \) is the half-distance between the source and sink, \( \eta = (\eta_o + \phi_o - \phi)/q \), and \( \xi = \phi/q \) are normalized variables, \( \eta_o \) is an equipotential encircling the input well, and \( \phi_o \) is the dipole source strength.

The problem is illustrated in Fig. 7 for \( q = d = 1 \), \( \phi_o = 10 \), \( \eta_o = -\sinh^{-1} 100 \). Only one-half of the flow domain is modeled due to symmetry. Cutting lines follow the vertical axis and form two small arcs of 0.01 radius around the source and sink. Using a 68-node discretization, of which only three nodes are located on each of the arcs, the computed flownet is visually identical to the analytical solution. Travel times computed along the stream lines, \( \psi = 0.4, 0.8, 1.2, \pi/2, 2, 2.4, \) and 2.8, at times, \( \tau = 0.015, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, \) and 0.6, are shown. The simulation results are indistinguishable from the analytical solution. The error in position norm \( ||e|| \) is shown as Table 1, where

\[
||e|| = \sqrt{(x - x')^2 + (y - y')^2}
\]

where \( [x,y] \) is the CVBEM-calculated location of the specified travel time along the flowpath, and \( [x',y'] \) is the position determined from the analytic solution. Note that in all cases the error norm is less than 0.01.

It was found that computed travel times to reach a target-value along the same streamline are identical regardless of the number of increments taken. This is partially attributed to the employment of quadrature with successive interval-halving. It should be mentioned that the emphasis here was placed on examining the applicability of the complex potential strategy and no effort was made to improve the numerical precision. The model accuracy could have been improved by allocating nodes along the boundary more appropriately.

Table 1
CVBEM position error norm, \( ||e|| \), at specified travel times, \( \tau \), and streamlines, \( \psi \).

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>( \pi/2 )</th>
<th>2</th>
<th>2.4</th>
<th>2.7</th>
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<td>0.0009</td>
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8. Summary and discussion

A strategy is proposed that uses the CVBEM to determine flow nets and travel times. Unlike finite-difference and finite-element approaches where volume discretization is required, boundary integral methods define the unknown variables only along the flow-domain boundary. The resulting number of nodes and corresponding system of equations is much smaller.

Disadvantages of the boundary-element method include the inability to directly incorporate unsteady flow conditions, to account for material heterogeneities within each flow domain, and for applications involving nonlinear equations. Another disadvantage lies in the need for solving the linear system of equations using gaussian elimination, as opposed to finite-difference and finite-element solvers which are more efficient.

The complex variable approach allows for the simultaneous solution of both fluid potentials and streamlines. The solution for any flow problem becomes one of finding the best values of complex nodal potentials that minimize the error between computed and assigned boundary conditions. OLS is used here to find this minimum error solution. While this approach may not be the most efficient, OLS provides the best linear unbiased estimate of unknown coefficients [11].

We present a strategy that employs CVBEM to provide the coordinates \( z = x + iy \) for any given set of arbitrary complex potentials \( \psi = \phi + i\psi \). Once all complex boundary potentials have been found, a reverse-mapping strategy is employed to find the flownet position by using complex boundary locations and potentials to find the complex position of specified internal streamline–equipotential intersections. The method avoids the need for interpolation during contour plotting. This approach accurately maps the complex potential domain into the complex position domain, except for cases where one-to-one correspondence fails.

One-to-one correspondence fails when multiple capture zones are present, leading to multiple locations where the specified complex potential may be found. This significant shortcoming of the technique can be avoided by using a streamline tracking algorithm [4]. Overlapping potentials are readily removed when multiple capture zones are present within the physical plane, causing branch-cuts on the complex potential. Flownets for each capture zone are resolved once individual capture zones have been delineated.

Travel times are determined by using increments in potential to determine the cumulative travel time along a streamline. This approach provides an accurate and flexible method for travel-time determination.

Additional improvements in accuracy and efficiency for determining both the unknown boundary potentials and travel times could be obtained by using improved boundary interpolators as well as more efficient computational methods. Faster computational methods were not necessary for the applications presented here due to the limited number of nodes, but may become necessary for larger problems.

The strategy is useful for flowfield visualization, solute front movement tracking, and calculation of arrival times at specified locations. Examples are presented for flow under a dam with a sheet pile, free surface within a dam with a seepage face, regional flow to a stream, and dipole solute front migration. Computed travel paths and travel times compare favorably with an analytic solution.

Acknowledgements

Fruitful discussions with Mark Bakker at the University of Georgia along with comments by anonymous reviewers substantially improved this manuscript.

References