Confined Aquifer

The PDE for unsteady ground-water flow is:

\[
T \left[ \frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right] = S \frac{\partial s}{\partial t} \tag{1}
\]

\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{1}{D} \frac{\partial s}{\partial t} \tag{2}
\]

\[
s'' + \frac{s'}{r} = \frac{s}{D} \tag{3}
\]

where \(D = T/S\), \(s'' = \frac{\partial^2 s}{\partial r^2}\), \(s' = \frac{\partial s}{\partial r}\), and \(s = \frac{\partial s}{\partial t}\).

The Theis solution to this equation for a constant pumping rate \(Q\) is:

\[
s = \frac{Q}{4\pi T} W(u) \tag{4}
\]

\[
u = \frac{r^2 S}{4T} \tag{5}
\]

\[
W(u) = \int_u^\infty \frac{e^{-x}}{x} \, dx \tag{6}
\]

**Derivative curve.** The Theis solution is a step response. The time derivative of the Theis solution is the impulse response:

\[
\dot{s} = \frac{\partial s}{\partial t} = \frac{\partial s}{\partial u} \frac{\partial u}{\partial t} \tag{7}
\]

\[
\frac{\partial s}{\partial u} = \frac{Q}{4\pi T} e^{-x} \bigg|_{x=u} = -\frac{Q}{4\pi T} \frac{e^{-u}}{u} \tag{8}
\]

\[
\frac{\partial u}{\partial t} = -\frac{r^2}{4Dt^2} = -\frac{u}{t} \tag{9}
\]

\[
\dot{s} = \frac{Q}{4\pi T} \frac{e^{-u}}{t} \tag{10}
\]

The maximum of this function is found by setting the second (time) derivative equal to zero:

\[
\ddot{s} = \frac{\partial^2 s}{\partial t^2} = \frac{Q}{4\pi T} \frac{e^{-u}}{t^2} (u - 1) = 0 \tag{11}
\]

which occurs when \(u = 1\). The peak time, \(t^*\), \(s^* = s(t^*)\), and \(\dot{s}^* = \dot{s}(t^*)\) can be used to determine \(D\) and \(T\) (note that \(S = T/D\)):

\[
D = \frac{r^2}{4ut^*} = \frac{r^2}{4t^*} \tag{12}
\]

\[
T = \frac{W(1) Q}{4\pi s^*} = \frac{Q}{57.28 s^*} \tag{13}
\]

\[
T = \frac{e^{-1} Q}{4\pi \dot{s}^* t^*} = \frac{Q}{34.16 \dot{s}^* t^*} \tag{14}
\]
Leaky Aquifer

Unsteady flow. The PDE for unsteady ground-water flow with leakage is:

\[
T \left[ \frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right] = S \frac{\partial s}{\partial t} + \frac{s}{c} \tag{15}
\]

\[
s'' + \frac{s'}{r} = \frac{\dot{s}}{D} + \frac{s}{T^2} \tag{16}
\]

where \(L^2 = Tc\). The analytic solution for this is:

\[
s = \frac{Q}{4\pi T} \int_u^\infty \frac{e^{-x-r^2/4\pi L^2}}{x} \, dx \tag{17}
\]

and the derivative curve is:

\[
\dot{s} = \frac{Q}{4\pi T} e^{-u-r^2/4\pi L^2} \frac{e^{-u}}{t} = \left[ \frac{Q}{4\pi T} \frac{e^{-u}}{t} \right] e^{-t/cS} \tag{18}
\]

Note that the term inside the brackets is the same as Equation 10.

Figure 1: Derivative curves for confined (Theis) and leaky (Walton) aquifers for unit parameters \((Q = T = S = L = r = 1)\).
Steady flow. The PDE for steady ground-water flow with leakage is:

\[ T \left[ \frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right] = \frac{s}{c} \]  \hspace{1cm} (19)

\[ s'' + \frac{s'}{r} = \frac{s}{L^2} \]  \hspace{1cm} (20)

This form is similar to the Modified Bessel Function Differential Equation (Abramowitz and Stegun, 9.6.1):

\[ z^2 w'' + z w' - \left( z^2 + v^2 \right) w = 0 \]  \hspace{1cm} (21)

for conditions \( v = 0, s = w, \) and \( z = r/L. \) The boundary conditions are:

\[ \lim_{z \to \infty} w = 0 \]  \hspace{1cm} (22)

\[ \lim_{z \to 0} z \frac{\partial w}{\partial z} = 1 \]  \hspace{1cm} (23)

which yields the analytic solution:

\[ w = K_0(z) = \int_{0}^{\infty} \frac{\cos zt}{\sqrt{t^2 + 1}} dt = \int_{0}^{\infty} \cos(z \sinh t) \, dt \]  \hspace{1cm} (24)

where \( K_0 \) is the modified Bessel function of the second kind.

Our boundary conditions are:

\[ \lim_{r \to \infty} s = 0 \]  \hspace{1cm} (25)

\[ \lim_{r \to 0} r \frac{\partial s}{\partial r} = \frac{Q}{2\pi T} \]  \hspace{1cm} (26)

which yields the analytic solution:

\[ s = \frac{Q}{2\pi T} K_0(r/L) \]  \hspace{1cm} (27)

Leakage into aquifers during well tests often result in steady flow conditions, which is commonly quantified using the specific capacity, \( C_p: \)

\[ C_p = \frac{Q}{s} = \frac{2\pi T}{K_0(r/L)} \]  \hspace{1cm} (28)
Sensitivity coefficients. Confined aquifer sensitivity coefficients are used to relate changes in drawdown to the aquifer parameters that determine drawdown, as well as how the parameters are related to each other.

\[
\frac{\partial s}{\partial t} = \frac{Q}{4\pi T} \frac{e^{-u}}{t} \tag{29}
\]

\[
\frac{\partial s}{\partial r} = \frac{Q}{4\pi T} \frac{2e^{-u}}{r} \tag{30}
\]

\[
\frac{\partial s}{\partial T} = \frac{Q}{4\pi T} \frac{e^{-u}W(u)}{T} \tag{31}
\]

\[
\frac{\partial s}{\partial S} = -\frac{Q}{4\pi T} \frac{e^{-u}}{S} \tag{32}
\]

\[
\frac{\partial s}{\partial D} = \frac{Q}{4\pi T} \frac{e^{-u}}{T} \tag{33}
\]

\[
\frac{\partial D}{\partial T} = \frac{e^{-u} - W(u)}{Se^{-u}} \tag{34}
\]

\[
\frac{\partial S}{\partial T} = \frac{W(u) - e^{-u}}{De^{-u}} \tag{35}
\]

\[T\text{ and } S\text{ are independent of each other only if:}\]

\[
\frac{\partial S}{\partial T} = 0 \tag{36}
\]

which occurs when \(W(u) = e^{-u}\). \(T\) and \(D\) are independent for these same conditions. In fact, drawdown is insensitive to \(T\) at this point, because \(\partial s/\partial T = 0\). This condition is satisfied at \(u = 0.4348\).

Exercises.

1. Plot the leaky aquifer derivative curves for a range of \(L\) values.
2. Rewrite the confined and leaky solutions using dimensionless variables.
3. Plot the sensitivity coefficients as a function of a) time \((t)\), b) distance \((r)\), and c) dimensionless well function argument \((u \text{ or } 1/u)\).