

Appendix A

Mathematical Hydrology

A.1 Units

Any quantitative subject requires the use of a measurement system, but deciding which system to use is a universal problem! Here's two...

English units. The English system that we commonly use derives from an archaic system of measurement:

Weights, Mass, Work:

1 dry (avoirdupois) ounce	=	437.5 grains
1 pound	=	16 dry ounces = 7,000 grains
1 stone	=	14 pounds
1 slug (mass)	=	32.17 pounds
1 quarter	=	25 pounds
1 hundredweight	=	4 quarters
1 ton	=	20 hundredweight
1 horsepower	=	550 ft-lbs/s

Length and Area:

1 yard	=	3 feet	=	36 inches
1 rod	=	25 links	=	16.5 ft
1 chain	=	4 rods	=	66 ft
1 furlong	=	10 chains	=	660 ft
1 mile	=	8 furlongs	=	5,280 ft
1 league	=	3 miles	=	15,840 ft
1 rood	=	10 rods × 1 chain	=	10,890 ft ²
1 acre	=	1 furlong × 1 chain	=	43,560 ft ²

Volume:

1 tablespoon	=	3 teaspoons = 8 drams
1 fluid ounce	=	2 tablespoons
1 gill	=	4 fluid ounces
1 cup	=	2 gills
1 pint	=	2 cups
1 quart	=	2 pints
1 gallon	=	4 quarts
1 peck	=	2 gallons
1 kenning	=	2 pecks
1 bushel	=	2 kennings
1 strike	=	2 bushels
1 pail	=	2 strikes
1 chaldron	=	4 pails
1 firkin	=	9 gallons
1 kilderkin	=	2 firkins
1 barrel	=	2 kilderkin
1 hogshead	=	3 kilderkin

These conversions often depend on the type of good being measured, however, and may not be universally valid.

Metric units. While English units are still used in a few countries, the international metric system is used by most nations, and virtually all scientists¹. The international *metric* system of units was established to standardize units of measure. Metric units are intended to be universal - independent of arbitrary human measurements. The meter was initially defined as a millionth of the distance from the north pole to the equator, measured along the longitude passing through Paris, France. The gram was defined as the mass of 1 mL (cm³) of water at its maximum density.

Metric Units:

Mass	kilogram	<i>kg</i>	
Length	meter	<i>m</i>	
Area	hectare	<i>ha</i>	
Volume	liter	<i>L</i>	
Time	second	<i>s</i>	
Energy	joule	<i>J</i>	<i>kg · m² / s²</i>
Power	watt	<i>W</i>	<i>J / s</i>
Force	newton	<i>N</i>	<i>J / m</i>
Pressure	pascal	<i>Pa</i>	<i>J / m³</i>

Metric Prefixes:

d	deci	10 ⁻¹	D	deca	10 ¹
c	centi	10 ⁻²	h	hecto	10 ²
m	milli	10 ⁻³	k	kilo	10 ³
μ	micro	10 ⁻⁶	M	mega	10 ⁶
n	nano	10 ⁻⁹	G	giga	10 ⁹
p	pico	10 ⁻¹²	T	tera	10 ¹²
f	femto	10 ⁻¹⁵	P	peta	10 ¹⁵
a	atto	10 ⁻¹⁸	E	exa	10 ¹⁸

Prefixes. Prefixes were also devised to allow rapid conversions from large to small units. For example, 1 *km* = 1,000 *m*. A problem with the metric prefix arises when

¹All countries in the world (except for Liberia, Burma/Myanmar, and the United States) use the metric system for official measurements. Unfortunately, mixing English with other units can result in engineering failures, such as the destruction of the \$125 million Mars Climate Orbiter spacecraft in 1999. In this case, two teams of scientists used different units, which were never reconciled. Also, the adoption of the newer *mks* (meter-kilogram-second) system to replace the older *cgs* (centimeter-gram-second) system has not been widely accepted.

there is an exponent on the unit. Does $1 \text{ km}^3 = 1 \text{ k}(m^3)$ = 1000 m^3 or does it equal $1 \text{ (km)}^3 = 1,000,000,000 \text{ m}^3$?

Significant Digits. Significant digits are used to convey the accuracy implicit in a number; $31,345.78 \pm 0.01$ is a much more precise number than $31,000 \pm 1,000$. The number of non-zero digits in the first value is much greater than the second. One would not report $31,345.78 \pm 1,000$ because all the non-zero digits after the first two are meaningless. For example, the number 3.1 has two digits (as does 31,000), while 3.14 has three and 3.14159 has six, and 3.141592653589793 has 16 digits.

When reporting results, the number of digits to display must recognize the accuracy of the digits in the calculation. For example, $1/\pi = 0.318309886183791$ has fifteen digits, while $1/3.1 = 0.322580645161290$ falsely conveys that the accuracy is still to 15 digits, when in fact the accuracy is only to the first two digits, i.e., $1/3.1 = 0.32$, because the value of π is only written to two digits.

This holds true regardless of the decimal place, i.e., $1/31 = 0.032$, or $1/0.31 = 3.2$, $1/0.031 = 32$. For two numbers of varying accuracy, the reported number of digits is limited by the number with the fewest digits, i.e., $\pi/3.1 = 1.0$, rather than 1.013416985028966 .

Yet, the digits should be preserved as much as possible when making calculations, because rounding during a calculation can reduce the accuracy of the final result. The final result should be simplified to the appropriate level of accuracy.

Problems

1. Convert 100 cfs (ft^3/s) into L/day . Use the appropriate prefix.
2. If water is flowing at a rate of 1 L/s over a drop of 10 m , what is the power production in watts? What is the total production (i.e., work or energy) in a year in $kW-hr$? In Joules?
3. If the ideal gas law is $PV = nRT$, what is the left hand side in terms of energy?
4. If 1 $ha = 10,000 \text{ m}^2$ and a hectare is 100 ares (the *hecto* prefix means that you multiply by 100), what is the area of an *are* in square meters?
5. What is the cost in \$ per acre-foot to lift water 100 feet using a pump efficiency of 60% and an electrical cost of \$0.05 per kW-hr?
6. What is 2.33×4.178503 ?

A.2 Transcendental Functions

Transcendental functions include the logarithmic, exponential, and trigonometric functions, such as sines and cosines. Table ?? summarizes these functions.

Table A.1: Common transcendental functions.

$\log x = \log_{10} x$
$\ln x = \log_e x = 2.3 \log x$
$10^x = 2.3e^x$
$\ln(u \cdot v) = \ln u + \ln v$
$\ln(u/v) = \ln u - \ln v$
$\ln(u^c) = c \ln u$
$e^{u+v} = e^u \cdot e^v$
$e^{u-v} = e^u / e^v$
$e^{u^c} = (e^u)^c$
$\sin \alpha = y/r$
$\cos \alpha = x/r$
$\tan \alpha = y/x$
$\sin(-\alpha) = -\sin \alpha$
$\cos(-\alpha) = \cos \alpha$
$\tan(-\alpha) = -\tan \alpha$
$\csc \alpha = 1/\sin \alpha$
$\sec \alpha = 1/\cos \alpha$
$\cot \alpha = 1/\tan \alpha$
$\sin^2 \alpha + \cos^2 \alpha = 1$
$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
$\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$

Exponents. The exponent is used to indicate the repeated multiplication of a number:

$$x^3 = x \cdot x \cdot x \quad (\text{A.1})$$

A negative exponent indicates the reciprocal of the number:

$$x^{-3} = \frac{1}{x^3} \quad (\text{A.2})$$

Non-integer exponents are also possible. For example, the exponent one-half indicates the square root of a number:

$$x^{1/2} = \sqrt{x} \quad (\text{A.3})$$

and

$$x^{1/n} = \sqrt[n]{x} \quad (\text{A.4})$$

A general rule for exponents is:

$$x^{a/b} = \sqrt[b]{x^a} = (\sqrt[b]{x})^a \quad (\text{A.5})$$

There are two standard *bases* for exponents. One is the base-10 system, such that any number can be represented using:

$$x = 10^b \quad (\text{A.6})$$

For example, $b = 2$ represents the number $10^2 = 100$. A second base is the natural system, represented using e :

$$x = e^a \quad (\text{A.7})$$

where $e \approx 2.71828$.

Logarithms. The logarithm of a number is equal to the exponent:

$$\log x = \log_{10}(10^b) = b \quad (\text{A.8})$$

$$\ln x = \log_e x = \log_e(e^a) = a \quad (\text{A.9})$$

There are several basic rules of exponentials and logarithms:

$$x \cdot y = 10^b \cdot 10^c = 10^{b+c} \quad (\text{A.10})$$

$$\frac{x}{y} = \frac{10^b}{10^c} = 10^{b-c} \quad (\text{A.11})$$

These relationships also hold for the natural system:

$$x \cdot y = e^b \cdot e^c = e^{b+c} \quad (\text{A.12})$$

$$\frac{x}{y} = \frac{e^b}{e^c} = e^{b-c} \quad (\text{A.13})$$

Logarithms also follow similar rules:

$$\log(b \cdot c) = \log(b) + \log(c) \quad (\text{A.14})$$

$$\log\left(\frac{b}{c}\right) = \frac{\log(b)}{\log(c)} \quad (\text{A.15})$$

for both the base-10 and natural systems. Another logarithmic relationship is:

$$\log(ax^b) = \log(a) + b \log(x) \quad (\text{A.16})$$

Trigonometry. We use trigonometry routinely to calculate angles, heights, distances, and other geometric properties.

Tangent. This is the rise over the run, or slope, i.e.,

$$\tan \alpha = \frac{\Delta y}{\Delta x} \quad (\text{A.17})$$

Sine. This is the rise over the length of the hypotenuse of a triangle:

$$\sin \alpha = \frac{\Delta y}{\Delta r} \quad (\text{A.18})$$

where $\Delta r^2 = \Delta x^2 + \Delta y^2$ is the Pythagorean theorem.

Cosine. This is the run over the hypotenuse:

$$\cos \alpha = \frac{\Delta x}{\Delta r} \quad (\text{A.19})$$

Acute & Oblique Triangles. For triangles that do not meet at a perpendicular angle, the Pythagorean theorem is replaced with:

$$a^2 + b^2 = c^2 + 2ab \cos \gamma \quad (\text{A.20})$$

where c is the side opposite angle γ .

Problems

1. What is the range of e^x if the range of x is $-\infty < x < \infty$?
2. What is the range of x if the range of $\log(x)$ is $-\infty < \log(x) < \infty$?
3. What is the pH of a 10 M HCl solution?
4. What is the $[OH]$ concentration if the $pH = 12.5$?
5. What is the normality of $pH = -3.6$ solution?
6. Write Darcy's Law using a trigonometric function.
7. How much longer is the slope length (i.e., hypotenuse) compared to the horizontal distance (i.e., run) for a slope of one percent? A slope of 45 percent?

A.3 Algebraic Equations

We use algebraic equations to show relationships between variables. For example,

$$y = c_o + c_1 x \quad (\text{A.21})$$

establishes a relationship between x and y where c_o is the intercept and c_1 is the slope. This is called a linear relationship between x and y . One can solve this equation for x in terms of y , using:

$$x = \frac{y - c_o}{c_1} \quad (\text{A.22})$$

A non-linear equation would take the form:

$$y = ax^b \quad (\text{A.23})$$

which can be linearized using $Y = \log y$ and $X = \log x$, so that:

$$Y = \log a + bX \quad (\text{A.24})$$

where the basic rules of logarithms are employed. One can solve for x using:

$$x = \left(\frac{y}{a}\right)^{1/b} \quad (\text{A.25})$$

Another nonlinear equation is:

$$y = c_o + c_1 x + c_2 x^2 \quad (\text{A.26})$$

which is a quadratic (second-order) polynomial. This can be linearized using:

$$y = c_o + c_1 X + c_2 X^2 \quad (\text{A.27})$$

where $X = x^2$ is new variable.

The quadratic equation can also be written in the form:

$$y = a + bx + cx^2 \quad (\text{A.28})$$

which can be solved in terms of x using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\text{A.29})$$

Summations. The summation operator, \sum , represents a repeated sum from the lower limit to the upper limit:

$$Y = \sum_{i=0}^2 y_i = y_0 + y_1 + y_2 \quad (\text{A.30})$$

or, in general:

$$Y = \sum_{i=0}^n y_i = y_0 + y_1 + y_2 + \dots + y_n \quad (\text{A.31})$$

Like the summation operator, the repeated multiplication operator is written as:

$$Y = \prod_{i=0}^2 y_i = y_0 \cdot y_1 \cdot y_2 \quad (\text{A.32})$$

$$Y = \prod_{i=0}^n y_i = y_0 \cdot y_1 \cdot y_2 \cdot \dots \cdot y_n \quad (\text{A.33})$$

Factorials. An *integer factorial* is defined using:

$$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1 \quad (\text{A.34})$$

For example,

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \quad (\text{A.35})$$

We use the *Gamma function*, $\Gamma(x)$, for *non-integer factorials*:

$$n! = \Gamma(n+1) \quad (\text{A.36})$$

For example, $6! = \Gamma(7) = 720$, and $5.5! = \Gamma(6.5) = 288$.

Convolution. We use convolution along with transfer functions to incorporate a delay between an input and an output:

$$y(t) = h_0 x(t) + h_1 x(t-1) + \dots + h_m x(t-m) \quad (\text{A.37})$$

which is the same as:

$$y(t) = \sum_{\tau=0}^m h(\tau) x(t-\tau) = h * x \quad (\text{A.38})$$

where m is the memory or maximum time lag of the system, and $*$ is the convolution operator.

The h function is called the *unit response function*, and corresponds to the response for a *Dirac* input:

$$y(t-t_a) = h(\tau) \quad (\text{A.39})$$

when $x(t) = \delta(t_a)$, where:

$$\delta(t_a) = \begin{cases} 0 & t < t_a \\ \infty & t = t_a \\ 0 & t > t_a \end{cases} \quad (\text{A.40})$$

This is a *spike* input.

The spike input can be summed to make a *step* input, $H(t_a)$:

$$H(t_a) = \int_0^\infty \delta(t_a) dt = \begin{cases} 0 & t < t_a \\ 1 & t \geq t_a \end{cases} \quad (\text{A.41})$$

which is also called the *Heaviside* function.

A *Pulse* function can be constructed from two Heaviside functions:

$$P(t_a, t_b) = H(t_a) - H(t_b) = \begin{cases} 0 & t < t_a \\ 1 & t_b > t \geq t_a \\ 0 & t \geq t_b \end{cases} \quad (\text{A.42})$$

Problems

1. Linearize the equation $y = a/x$.
2. Find an equation for a quadratic polynomial, $y = f(x)$, in the form $x = f(y)$. Can you linearize this function?
3. Write out the equation $y = \sum_{i=0}^3 c_i x^i$
4. Write the factorial equation using the repeated multiplication operator \prod .
5. Can you find a way to write the \prod operator in terms of the \sum operator? [Hint, does taking the logarithm help?]
6. What is $12! / (7! \times 5!)$?
7. What is $3.5! / (2.2! \times 1.3!)$?
8. Find $y(t)$ if $x(t) = [\dots, 0, 7, 15, 11, 0, \dots]$ and $h(\tau) = [1, 5, 4]$.

A.4 Matrices

A matrix is an organized set of numbers. For example, a simple, two-dimensional matrix may look like:

$$\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \quad (\text{A.43})$$

where **boldface** indicates that there are two subscripts, u_{ij} , which refer to the elements in each row and column, respectively. In this case, there are three rows and two columns.

We indicate the size of the matrix using the notation, (m, n) , where m is the number of rows, and n is the number of columns. For our case, the size is $(3, 2)$. A three-dimensional matrix would have a size of (m, n, p) , where p represents the number of layers in the matrix.

The *transpose* of a matrix reverses the orientation of the matrix. That is, the columns are exchanged with the

rows. Using the previous matrix as an example, the transpose of \mathbf{u} is:

$$\mathbf{u}^T = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \end{bmatrix} \quad (\text{A.44})$$

where the superscript T is used to indicate the transpose. The apostrophe, $'$, is another symbol that is commonly used to indicate the transpose, $\mathbf{u}' = \mathbf{u}^T$. The size of \mathbf{u}^T is (n, m) , or $(2, 3)$ for this example. Note that one only has to switch the location of the indices, $u_{ij}^T = u_{ji}$, to obtain the transpose.

Adding, subtracting, multiplying, and dividing by a constant, c , to a matrix is obtained by performing the operation to every element of the matrix:

$$c + \mathbf{u} = \begin{bmatrix} c + u_{11} & c + u_{12} \\ c + u_{21} & c + u_{22} \\ c + u_{31} & c + u_{32} \end{bmatrix} \quad (\text{A.45})$$

$$c\mathbf{u} = \begin{bmatrix} cu_{11} & cu_{12} \\ cu_{21} & cu_{22} \\ cu_{31} & cu_{32} \end{bmatrix} \quad (\text{A.46})$$

Matrices can be added and subtracted if they have the same size. That is, if they have the same number of rows and columns.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \\ u_{31} + v_{31} & u_{32} + v_{32} \end{bmatrix} \quad (\text{A.47})$$

where the size of both \mathbf{u} and \mathbf{v} is $(3, 2)$. Matrix addition is also possible if one matrix has a lower dimension. For example, we can perform the operation:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} + v_1 & u_{12} + v_1 \\ u_{21} + v_2 & u_{22} + v_2 \\ u_{31} + v_3 & u_{32} + v_3 \end{bmatrix} \quad (\text{A.48})$$

where the size of \mathbf{v} is $m = 3$. Note that it is also possible to add a vector with a size of $n = 2$ by taking the transpose:

$$\mathbf{u} + \mathbf{v}^T = \begin{bmatrix} u_{11} + v_1 & u_{12} + v_2 \\ u_{21} + v_1 & u_{22} + v_2 \\ u_{31} + v_1 & u_{32} + v_2 \end{bmatrix} \quad (\text{A.49})$$

Matrix multiplication is achieved by multiplying every row of the left matrix, \mathbf{u} , by every column of the right matrix, \mathbf{v} . In this case, the number of columns of the left matrix must equal the number of rows of the right matrix, and the result has a size equal to the number of rows of the left matrix and the number of columns of the right matrix:

$$(m, n) \times (n, p) = (m, p) \quad (\text{A.50})$$

$$\mathbf{u}\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (\text{A.51})$$

$$= \begin{bmatrix} u_{11}v_1 + u_{12}v_2 \\ u_{21}v_1 + u_{22}v_2 \\ u_{31}v_1 + u_{32}v_2 \end{bmatrix} \quad (\text{A.52})$$

For this problem, the multiplication is $(3, 2) \times (2, 1) = (3, 1)$.

The *identity matrix*, \mathbf{I} , is defined as the matrix which when multiplied by another matrix yields the original matrix:

$$\mathbf{u}\mathbf{I} = \mathbf{u} \quad (\text{A.53})$$

where the sizes of the \mathbf{u} and \mathbf{I} matrices are (m, n) and (n, n) , respectively. The identity matrix takes the form:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.54})$$

for $n = 3$. Note that \mathbf{I} is always a square matrix, where the diagonal elements all equal one and all the off-diagonal elements all equal zero.

Matrix division is not possible, but we can obtain the desired solution by using the matrix inverse, \mathbf{u}^{-1} :

$$\mathbf{w} = \mathbf{u}\mathbf{v} \quad (\text{A.55})$$

$$\mathbf{u}^{-1}\mathbf{w} = \mathbf{u}^{-1}\mathbf{u}\mathbf{v} = \mathbf{I}\mathbf{v} \quad (\text{A.56})$$

$$\mathbf{v} = \mathbf{u}^{-1}\mathbf{w} \quad (\text{A.57})$$

where $\mathbf{u}^{-1}\mathbf{u} = \mathbf{I}$. For a 2×2 matrix, the inverse is:

$$\mathbf{u}^{-1} = \frac{1}{\det \mathbf{u}} \begin{bmatrix} u_{22} & -u_{21} \\ -u_{12} & u_{11} \end{bmatrix} \quad (\text{A.58})$$

where $\det \mathbf{u}$ is the *determinant*, defined for a 2×2 matrix using:

$$\det \mathbf{u} = u_{11}u_{22} - u_{12}u_{21} \quad (\text{A.59})$$

which is just the area of the parallelogram with the two sides defined by the two rows of the matrix, or the volume of the parallelepiped for a 3×3 matrix.

Problems

1. Find the sum of $\mathbf{u} = [0, 1; 1, 3]$ and $\mathbf{v} = [3, 2; 6, 1]$.
2. Find the product of \mathbf{u} and \mathbf{v} .
3. Find the determinants of \mathbf{u} and \mathbf{v} .

A.5 Vectors

A vector is an oriented list of numbers. A vector with two elements would look like:

$$\vec{v} = [v_1, v_2] \quad (\text{A.60})$$

We can plot these elements on two perpendicular axes, such as the x - and y -axes of a cartesian plot. We say

these axes are *orthogonal*, or perpendicular, to each other. The vector is *oriented* because, like a pointer, it has a *base*, located at the origin, and the *tip*, located at the position indicated by the list of numbers.

Another way of expressing a vector mathematically is:

$$\vec{v} = [v_1, v_2] = \mathbf{i} v_1 + \mathbf{j} v_2 \quad (\text{A.61})$$

where the symbols \mathbf{i} and \mathbf{j} are just mathematical symbols to indicate the x - and y -axes. A three-dimensional vector would be:

$$\vec{v} = [v_1, v_2, v_3] = \mathbf{i} v_1 + \mathbf{j} v_2 + \mathbf{k} v_3 \quad (\text{A.62})$$

where there are now three axes, corresponding to the x -, y -, and z -directions. Higher-order vectors are also possible, although not commonly used.

Algebra. Vectors addition and subtraction are performed for each element, independent of the other elements:

$$\begin{aligned} \vec{u} + \vec{v} &= [u_1, u_2] + [v_1, v_2] \\ &= [u_1 + v_1, u_2 + v_2] \end{aligned} \quad (\text{A.63})$$

$$\begin{aligned} \vec{u} - \vec{v} &= [u_1, u_2] - [v_1, v_2] \\ &= [u_1 - v_1, u_2 - v_2] \end{aligned} \quad (\text{A.64})$$

These operations are associative and commutative:

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \quad (\text{A.65})$$

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad (\text{A.66})$$

Multiplication or division of a constant, c , is also performed element by element:

$$c \cdot \vec{u} = [cu_1, cu_2] \quad (\text{A.67})$$

$$\frac{\vec{u}}{c} = \left[\frac{u_1}{c}, \frac{u_2}{c} \right] \quad (\text{A.68})$$

Multiplication of two vectors is more complicated. In fact, there are two types of multiplication, the dot product, $\vec{u} \cdot \vec{v}$, and the cross product, $\vec{u} \times \vec{v}$. Two dimensional multiplication is defined using:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 \quad (\text{A.69})$$

$$\vec{u} \times \vec{v} = u_1 v_2 - u_2 v_1 \quad (\text{A.70})$$

Three dimensional multiplication yields:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad (\text{A.71})$$

$$\vec{u} \times \vec{v} = [u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1] \quad (\text{A.72})$$

which is a vector. Note that the dot-product yields a *scalar* (single value), while the cross-product yields a *vector*. Vector multiplication is commutative for the dot product, but not for the cross-product:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \quad (\text{A.73})$$

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \quad (\text{A.74})$$

Vector division is undefined.

Geometry. As noted above, a vector can be represented using a pointer, with the base of the pointer resting at the origin and the tip of the pointer resting at the value of the vector. Another way of representing the vector is to define the magnitude, r , of the vector using:

$$r = \text{abs}\{\vec{u}\} = \sqrt{u_1^2 + u_2^2} \quad (\text{A.75})$$

and a rotation angle, θ , using:

$$\theta = \text{arg}\{\vec{u}\} = \arctan \frac{u_2}{u_1} \quad (\text{A.76})$$

so that:

$$u_1 = r \cos \theta \quad (\text{A.77})$$

$$u_2 = r \sin \theta \quad (\text{A.78})$$

The addition of two vectors, $\vec{w} = \vec{u} + \vec{v}$, can be visualized by placing the base of the second vector at the tip of the first vector. The resulting vector is the pointer connecting the base of the first pointer to the tip of the second pointer.

$$\begin{aligned} r_w &= \text{abs}\{\vec{w}\} = \sqrt{w_1^2 + w_2^2} \\ &= \sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2} \end{aligned} \quad (\text{A.79})$$

$$\theta_w = \text{arg}\{\vec{w}\} = \arctan \frac{u_2 + v_2}{u_1 + v_1} \quad (\text{A.80})$$

The geometry of a three-dimensional vector is commonly written using two angles, θ and λ :

$$u_1 = r \cos \theta \cos \lambda \quad (\text{A.81})$$

$$u_2 = r \sin \theta \cos \lambda \quad (\text{A.82})$$

$$u_3 = r \sin \theta \sin \lambda \quad (\text{A.83})$$

Problems

1. Find the sum of $u = [6, 4]$ and $v = [10, 3]$.
2. Find the dot- and cross-products of u and v .
3. Find the magnitudes of u and v .

A.6 Complex Numbers

A complex variable is a number composed of two parts, one called the *real* part and a second called the *imaginary* part.

$$u = u_o + \mathbf{i} u_1 \quad (\text{A.84})$$

where \mathbf{i} is used to indicate the imaginary part. We denote the real and imaginary parts using the notation:

$$\Re\{u\} = u_o \quad (\text{A.85})$$

$$\Im\{u\} = u_1 \quad (\text{A.86})$$

The *complex conjugate* of an imaginary number is defined using:

$$\bar{u} = u_o - \mathbf{i} u_1 = \Re\{u\} - \Im\{u\} \quad (\text{A.87})$$

Algebra. Adding two complex variables is accomplished by adding the real and imaginary parts separately:

$$u + v = (u_o + \mathbf{i} u_1) + (v_o + \mathbf{i} v_1) \quad (\text{A.88})$$

$$= (u_o + v_o) + \mathbf{i} (u_1 + v_1) \quad (\text{A.89})$$

Subtraction is equivalently defined using:

$$u - v = (u_o + \mathbf{i} u_1) - (v_o + \mathbf{i} v_1) \quad (\text{A.90})$$

$$= (u_o - v_o) + \mathbf{i} (u_1 - v_1) \quad (\text{A.91})$$

Note that $u + \bar{u} = 2\Re\{u\}$ and $u - \bar{u} = 2\Im\{u\}$.

Multiplication and division with a real value, c , where $\Im(c) = 0$, is allowed:

$$cu = cu_o + \mathbf{i} cu_1 \quad (\text{A.92})$$

$$\frac{u}{c} = \frac{u_o}{c} + \mathbf{i} \frac{u_1}{c} \quad (\text{A.93})$$

Multiplication of two complex variables is achieved using term-by-term expansion:

$$uv = (u_o + \mathbf{i} u_1)(v_o + \mathbf{i} v_1) \quad (\text{A.94})$$

$$= (u_o v_o + \mathbf{i}^2 u_1 v_1) + \mathbf{i} (u_o v_1 + u_1 v_o) \quad (\text{A.95})$$

The convention is that whenever the imaginary part is multiplied by itself, the result is a real value with a negative sign, i.e.:

$$\mathbf{i}^2 = -1 \quad (\text{A.96})$$

so that:

$$uv = (u_o v_o - u_1 v_1) + \mathbf{i} (u_o v_1 + u_1 v_o) \quad (\text{A.97})$$

Note that any complex variable, u , multiplied by its complex conjugate, \bar{u} , yields:

$$u\bar{u} = (u_o + \mathbf{i} u_1)(u_o - \mathbf{i} u_1) \quad (\text{A.98})$$

$$= (u_o^2 + u_1^2) + \mathbf{i} (u_o u_1 - u_1 u_o) = \text{abs}\{u\}^2 \quad (\text{A.99})$$

which is a real number.

Division of complex numbers is not strictly possible, but we can get around this using the complex conjugate. Given that we know $w = uv$ and u , and wish to find v , we first multiply both sides by the complex conjugate of u , and then divide by their product:

$$w = uv \quad (\text{A.100})$$

$$\bar{u}w = \bar{u}uv \quad (\text{A.101})$$

$$v = \frac{\bar{u}w}{\bar{u}u} \quad (\text{A.102})$$

This is possible because division by a real value is allowed.

Geometry. The complex number, u , can be written using:

$$u = r(\cos \theta + \mathbf{i} \sin \theta) \quad (\text{A.103})$$

which is similar to the two-dimensional vector definition. It can also be shown that:

$$u = re^{\mathbf{i}\theta} \quad (\text{A.104})$$

where:

$$e^{\mathbf{i}\theta} = \cos \theta + \mathbf{i} \sin \theta \quad (\text{A.105})$$

Note that $e^{\mathbf{i}\pi} = -1$. Using the fact that multiplication of exponential functions is equivalent to the addition of the exponents yields:

$$w = uv = (r_u e^{\mathbf{i}\theta_u}) (r_v e^{\mathbf{i}\theta_v}) = r_u r_v e^{\mathbf{i}(\theta_u + \theta_v)} \quad (\text{A.106})$$

which means that $r_w = r_u r_v$ and $\theta_w = \theta_u + \theta_v$.

One can obtain a sense of what the imaginary number, i , represents by noting that multiplication is a rotation about the origin. Multiplying a vector by -1 represents a rotation of 180° about the origin.

A rotation of 360° involves two rotations of 180° , so the rotation requires two successive multiplications of -1 , or $(-1) \times (-1) = 1$, which places the vector back in its original position.

With this example in mind, note that a multiplication by \mathbf{i} results in a rotation of 90° . This is because two successive multiplications (or rotations) yields \mathbf{i}^2 , which is the same as 180° , which is just -1 , so that:

$$\mathbf{i} = \sqrt{-1} \quad (\text{A.107})$$

Problems

1. Find the sum of $u = 6 + 4\mathbf{i}$ and $v = 10 + 3\mathbf{i}$.
2. Find the product of u and v .
3. Find the value of u divided by v .
4. Find the magnitudes of u and v .

A.7 Calculus

An *ordinary* differential equation, ODE, describes how one or more variables change as a function of one other variable. For example, the velocity can be written as an ordinary differential equation - the change in position is a function of a single variable, time:

$$v = \frac{dx}{dt} \quad (\text{A.108})$$

and the acceleration is the second derivative:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad (\text{A.109})$$

Table A.2: General differentiation rules.

Function	Derivative
$u = f(x)$	$u' = du/dx$
c	0
cx	c
cu	$c u'$
x^c	$c x^{c-1}$
u^c	$c u^{c-1} u'$
uv	$u v' + v u'$
u/v	$(v u' - u v')/v^2$
$\ln u$	u'/u
$\exp u$	$u' \exp u$
$\sin u$	$u' \cos u$
$\cos u$	$-u' \sin u$

In a similar manner, water levels, h , may change with respect to time, t :

$$h = ct + b \quad (\text{A.110})$$

where c and b are constants. The derivative, or change in water levels, with respect to time would be:

$$\dot{h} = \frac{dh}{dt} = c \quad (\text{A.111})$$

If, instead of time, water levels change with respect to distance, x , we have:

$$h = mx + b \quad (\text{A.112})$$

where m and b are again constants. The derivative, in this case, would be:

$$h' = \frac{dh}{dx} = m \quad (\text{A.113})$$

A few basic differential rules are provided in Table ???. If water levels in more than one aquifer are observed, then we might have multiple derivatives with respect to time or position, i.e.,

$$\dot{h}_1 = m \dot{h}_2 \quad \text{or} \quad h_1' = m h_2' \quad (\text{A.114})$$

In both cases, the water levels only change as a function of time or position, but not both.

If the water level varies with both time and position, then we must write a *partial* differential equation, PDE:

$$\frac{\partial h}{\partial x} = m \frac{\partial h}{\partial t} \quad (\text{A.115})$$

where the partial derivative means that the variable is held constant for all other variables than the one used for taking the derivative. In this case, the left-hand side is calculated for a specific time, t_i while the right-hand side is calculated at a specific position, x_j :

$$(h')_{t_i} = m \left(\dot{h} \right)_{x_j} \quad (\text{A.116})$$

We can approximate derivatives using *finite differences*, that is:

$$u' = \frac{du}{dx} \approx \frac{\Delta u}{\Delta x} = \frac{u_b - u_a}{x_b - x_a} \quad (\text{A.117})$$

In fact, the limit as Δx goes to zero is just the definition of the derivative!

Integration is the reverse of differentiation. That is, the integral of the derivative just brings us back to the original function. It is the same as the summation, where the function is divided up into tiny pieces, and then added up:

$$h(x) = \int_o^x h' dx \approx \sum_{i=o}^x \Delta h(x) \quad (\text{A.118})$$

where Δh is the change in h . We can also integrate the time derivative:

$$h(t) = \int_o^t \dot{h} dt \approx \sum_{i=o}^t \Delta h(t) \quad (\text{A.119})$$

Vector Calculus. The derivative of spatially variable function, $U = f(x, y, z)$, yields a vector:

$$\begin{aligned} \vec{u} = \nabla U &= \left[\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right] = [u_1, u_2, u_3] \\ &= i u_1 + j u_2 + k u_3 \end{aligned} \quad (\text{A.120})$$

where ∇ is the gradient, or *grad*, operator. For two-dimensional problems, this would be:

$$\vec{u} = \nabla U = \left[\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right] = [u_1, u_2] = i u_1 + j u_2 \quad (\text{A.121})$$

Just as there are two types of vector multiplication, the dot (\cdot) and cross (\times) products, there are also two types of vector derivatives, which in two dimensions are:

$$\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \quad (\text{A.122})$$

$$\nabla \times \vec{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \quad (\text{A.123})$$

and for three-dimensional vectors:

$$\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \quad (\text{A.124})$$

$$\nabla \times \vec{u} = \left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right] \quad (\text{A.125})$$

Note that if $\vec{u} = \nabla U$, then

$$\nabla \cdot \vec{u} = \nabla \cdot \nabla U = \nabla^2 U \quad (\text{A.126})$$

$$\nabla \times \vec{u} = \nabla \times \nabla U = 0 \quad (\text{A.127})$$

because $\partial(\partial U/\partial x)/\partial y = \partial(\partial U/\partial y)/\partial x$, etc.

Table A.3: Vector calculus identities.

Vector Derivatives:

$$\nabla A = \vec{a} \quad (\text{Gradient, or Grad})$$

$$\nabla(AB) = A\vec{b} + B\vec{a}$$

$$\begin{aligned} \nabla(\vec{a} \cdot \vec{b}) &= (\vec{b} \cdot \nabla)\vec{a} + (\vec{a} \cdot \nabla)\vec{b} \\ &\quad + \vec{b} \times (\nabla \times \vec{a}) + \vec{a} \times (\nabla \times \vec{b}) \end{aligned}$$

$$\begin{aligned} \nabla^2 A &= \nabla \cdot \nabla A \quad (\text{Divergence, or Div}) \\ &= \nabla \cdot \vec{a} \end{aligned}$$

$$\begin{aligned} \nabla \cdot (A\vec{b}) &= \vec{b} \cdot \vec{a} + A\nabla \cdot \vec{b} \\ &= \vec{b} \cdot \vec{a} + A\nabla^2 B \end{aligned}$$

$$\begin{aligned} \nabla \times \nabla A &= \nabla \times \vec{a} \quad (\text{Curl}) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \nabla \times (A\vec{b}) &= \vec{a} \times \vec{b} + A(\nabla \times \vec{b}) \\ &= \vec{a} \times \vec{b} \end{aligned}$$

$$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) = 0$$

$$\nabla \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b} + (\nabla \cdot \vec{b})\vec{a} - (\nabla \cdot \vec{a})\vec{b}$$

Vector Integrals:

$$\begin{aligned} \int_V \vec{a} \, dV &= \int_V \nabla A \, dV \\ &= \int_S A \, dS \end{aligned}$$

$$\int_V (\nabla \cdot \vec{a}) \, dV = \int_S \vec{a} \cdot dS \quad (\text{Divergence theorem})$$

$$\int_V (\nabla \times \vec{a}) \, dV = - \int_S \vec{a} \times dS$$

$$\int_S (\nabla \times \vec{a}) \cdot dS = \oint_C \vec{a} \cdot dC \quad (\text{Stokes' theorem})$$

Problems

1. Find the derivative of $y = 3 + 2x + 5x^2$.
2. Find the partial derivatives of $y = 3 + 2x + 4t + 5xt$.
3. Find the integral of $y = 2 + 3x$ between 4 and 5.
4. Show that the curl of the gradient of a scalar field is always zero, i.e.,

$$\nabla \times \nabla H = \nabla \times \vec{h} = 0 \quad (\text{A.128})$$

so that $\nabla \times \vec{q} = 0$.

5. Show that Darcy's law can be written in terms of stream potentials A and B using:

$$\vec{q} = -K [\nabla \times (\vec{a}B)] = K [\nabla \times (A\vec{b})] \quad (\text{A.129})$$

where $\vec{a} \cdot \vec{h} = 0$ and $\vec{b} \cdot \vec{h} = 0$.